Relativistic Rocket

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I. INTRODUCTION

India's ISRO has done the country proud by successfully putting the research probes Chandrayaan-1 and Mangalyaan orbiting around the Moon and Mars, respectively. Data collected by Chandrayaan-1 gave the first concrete evidence for the presence of water on Moon. It also played a key role in discovering a cave-like tunnel just below the lunar surface.

Both of these probes were injected into pre-determined orbits by the PSLV rockets, which have indeed a splendid track record. As far as rocket fuel is concerned, these launch vehicles use liquid propellants like unsymmetrical dimethylhydrazine, with nitrogen tetroxide acting as the oxidizer.

Can we think of alternate compact but high efficiency rocket fuel? Now, any photon of frequency ν has energy $h\nu$ and momentum $h\nu/c$. So, can we speculate about rockets that are propelled by a collimated jet of high energy photons? These photons could be created by matter-antimatter annihilation that relies on the Einstein's result $E = mc^2$. In what follows, we will discuss the theory behind such a relativistic spacecraft.

II. SPECIAL RELATIVISTIC PROPULSION

Consider a specially designed rocket that propels itself forward by ejecting from its rear end nozzle a narrow beam of intense gamma-ray photons, created due to matter-antimatter annihilation taking place in the fuel chamber. The fuel chamber can have two magnetic bottles like parts - one containing e.g. protons, and the other, anti-protons. We will now go through a detailed calculation in order that sometime in the future, perhaps such a rocket will be designed based on this analysis.

Suppose the matter-antimatter annihilation begins at time $t = 0$, as measured by clocks on Earth, when the rocket was initially at rest on the surface of the Earth. We will assume that Earth is an inertial frame and we will also neglect Earth's gravity throughout the calculation. In the frame of the rocket, the matter-antimatter annihilation proceeds in such a way as to decrease the rest mass of the rocket at the rate,

$$\frac{dM}{d\tau} = -\, \alpha M \tag{1}$$

where α is a positive constant, and τ is the proper time in the rocket's frame.

Assuming that at Earth time $t = 0$ (and also, proper time $\tau = 0$ in the rocket's frame), the rest mass of the rocket is M_0, the solution to eq.(1) is given by,

$$\int_{M_0}^{M} \frac{dM}{M} = -\alpha \int_0^{\tau} d\tau \tag{2}$$

so that,

$$M(\tau) = M_0 \exp(-\alpha\tau) \tag{3}$$

We can estimate the rates of energy $\frac{dE_{photons}}{d\tau}$ and momentum $\frac{dp_{photons}}{d\tau}$ carried away by the gamma-ray photons, as measured by an observer in the rocket as follows.

In the rest frame of the rocket, production of photons is at the expense of decrease in the mass of the rocket due to annihilation, so that,

$$\frac{dE_{photons}}{d\tau} = -c^2 \frac{dM}{d\tau} = c^2\, \alpha M \tag{4}$$

Since, photons with energy E carry momentum E/c, rate of change of photon momentum is,

$$\frac{dp_{photons}}{d\tau} = -\frac{1}{c} \frac{dE_{photons}}{d\tau} = -c\, \alpha M \tag{5}$$

where the -ve sign appears due to the fact that photon momenta are in the -ve x direction.

Because of the powerful gamma photon thrust, the rocket begins accelerating in the +ve x-direction starting at Earth time $t = 0$ (corresponding to the proper time instant $\tau = 0$ in the rocket's frame). At Earth time t, when the rocket is traveling with an instantaneous velocity $v(t)$ along the +ve x-direction, its proper time interval $d\tau$ is related to Earth's time interval dt as,

$$d\tau = \sqrt{1 - v^2(t)/c^2}\; dt\ . \tag{6}$$

Hence, using eq.(6) in eqs.(4) and (5), the rate at which the energy and momentum of the rocket is changing with time as measured by an observer on Earth is given by,

$$\frac{dE_{roc}}{dt} = -\alpha c^2 M(t)\left[1 - \frac{v(t)}{c}\right] \tag{7}$$

and,

$$\frac{dp_{roc}}{dt} = \alpha c M(t)\left[1 - \frac{v(t)}{c}\right]\ , \tag{8}$$

respectively, where $M(t)$ is the rest mass of the rocket at Earth time t.

Since energy E' and momentum p' as measured by an observer on Earth is related to energy E and momentum p as measured by an observer on the rocket as,

$$E' = \frac{E + vp}{\sqrt{1 - v^2/c^2}} \tag{9}$$

and,

$$p' = \frac{p + vE/c^2}{\sqrt{1 - v^2/c^2}}\ , \tag{10}$$

the rates of energy and momentum carried by gamma ray photons as measured by an Earth based observer is given by,

$$\frac{dE'_{photons}}{dt} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{d\tau}{dt}\left[\frac{dE_{photons}}{d\tau} + v\frac{dp_{photons}}{d\tau}\right]$$

$$= \alpha c^2 M(t)\left[1 - \frac{v(t)}{c}\right] \tag{11}$$

and,

$$\frac{dp'_{photons}}{dt} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{d\tau}{dt}\left[\frac{dp_{photons}}{d\tau} + \frac{v}{c^2}\frac{dE_{photons}}{d\tau}\right]$$

$$= -\alpha c M(t)\left[1 - \frac{v(t)}{c}\right] \tag{12}$$

And because total energy and total momentum of (rocket + gamma ray photons)-system is conserved in the Earth's frame, respectively, one has,

$$\frac{dE_{roc}}{dt} = -\frac{dE'_{photons}}{dt} = -\alpha c^2 M(t)\left[1 - \frac{v(t)}{c}\right] \tag{13}$$

and,

$$\frac{dp_{roc}}{dt} = -\frac{dp'_{photons}}{dt} = \alpha c M(t)\left[1 - \frac{v(t)}{c}\right] \tag{14}$$

Employing the standard relation between rest mass, speed and momentum, we have in this case,

$$p_{roc} = \frac{M(t)v(t)}{\sqrt{1-v^2/c^2}} \ , \tag{15}$$

By differentiating eq.(15) with respect to t, we obtain,

$$\frac{dp_{roc}}{dt} = \frac{dM(t)}{dt}\frac{v(t)}{\sqrt{1-v^2/c^2}} + M(t)\frac{d}{dt}\left[\frac{v(t)}{\sqrt{1-v^2/c^2}}\right] \tag{16}$$

Making use of eqs.(1) and (6), we find that,

$$\frac{dM(t)}{dt} = \frac{d\tau}{dt}\frac{dM(\tau)}{d\tau} = \sqrt{1-v^2/c^2}\frac{dM(\tau)}{d\tau} = -\alpha M(t)\sqrt{1-v^2/c^2} \tag{17}$$

Substituting the result of eq.(17) in eq.(16) we get,

$$\frac{d}{dt}\left[\frac{v(t)}{\sqrt{1-v^2/c^2}}\right] = \alpha c \tag{18}$$

The above equation can be trivially integrated so that,

$$\frac{v(t)}{\sqrt{1-v^2/c^2}} = \alpha c t \tag{19}$$

leading to,

$$v(t) = \frac{\alpha c t}{\sqrt{1+(\alpha t)^2}} \tag{20}$$

Although it appears from eq.(20) that as $t \to \infty$ one has $v \to c$, one should realize that the rocket to begin with has a bare rocket mass, an observer and equal amount of fuel (matter + antimatter). Once the fuel gets completely annihilated at the Earth time t_f, there is no more acceleration and thereafter, the rocket + observer cruise ahead with a subluminal constant velocity,

$$v(t_f) = \frac{\alpha c t_f}{\sqrt{1+(\alpha t_f)^2}} \tag{21}$$

It is interesting to ask: how does the angular diameter θ of Earth as measured by an observer in the rocket change with time? In the limit $t \to \infty$, what happens to θ?

At Earth's time $t < t_f$, the distance traveled by the rocket as measured by an observer on Earth is,

$$d(t) = \int_0^t v(t')dt' \tag{21}$$

so that the distance as measured in the rocket's frame will be Lorentz-contracted to,

$$d_{roc}(t) = \sqrt{1-v(t)^2/c^2}\ \ d(t) = \frac{c}{\alpha}\left[1 - \frac{1}{\sqrt{1+(\alpha t)^2}}\right] . \tag{22}$$

Hence, the angular size of the Earth as measured by an observer on the rocket for $t < t_f$ is given by,

$$\theta(t) = \frac{R_E}{d_{roc}(t)} = \frac{\alpha R_E}{c}\left[1 - \frac{1}{\sqrt{1+(\alpha t)^2}}\right]^{-1} . \tag{23}$$

For $t > t_f$, we will have the distance as measured in rocket's frame to be,

$$D_{roc}(t) \equiv \sqrt{1-v(t_f)^2/c^2}\left[v(t_f)(t-t_f) + \int_0^{t_f} v(t)dt\right]$$

$$= \frac{\alpha c t_f(t-t_f)}{1+(\alpha t)^2} + \frac{c}{\alpha}\left[1 - \frac{1}{\sqrt{1+(\alpha t_f)^2}}\right] . \tag{24}$$

Hence, for $t > t_f$, the angular size is given by,

$$\theta(t) = \frac{R_E}{D_{roc}(t)} \tag{25}$$

In the limit, $t \to \infty$, since $D_{roc} \to \infty$, we get $\theta(t) \to 0$.

III. CONCLUSIONS

The Kepler telescope of NASA has been discovering exoplanets at a very rapid rate. It is natural therefore that every one fantasizes of space voyage, hopping from one extra-solar planetary system to another. But since the nearest star (other than the Sun) is more than 3 light-years away, one needs space-crafts to travel with speeds faster than $\sim 0.1\ c$.

The purpose of this article is to describe an interesting theoretical application of special relativity in the field of rocket launches based on action-reaction principle. The present article demonstrates that if one uses propulsion which relies on matter-antimatter annihilation, one can attain very high speeds. Of course, the study described here is purely theoretical, and does not address severe technological limitations of space travel.

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